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Boundary integral equation technique with application to freezing around a buried pipe

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Abstract—In this paper the use of the boundary integral equation method (BIEM) for multidimensional problems with a moving phase-change interface is explored. The method is shown to be suited to heat transfer problems where the field equations are linear in each region and the boundary or interface matching conditions are both highly irregular and non-linear. For moving interface problems the BIE technique both reduces the dimensions of the problem by one, thus decreasing storage requirements, and directly solves for the unknown normal temperature gradient on each side of the interface for the determination of the instantaneous interface velocity. To illustrate the versatility of this technique the BIEM is applied to a previously unsolved problem: the melting/freezing around a pipe buried in a semi-infinite domain where the melting/freezing is initiated at the free surface and the medium is initially not at the phase-change temperature. For simplicity quasi-steady heat conduction is assumed in both the thawed and frozen zones. Solutions are presented for various values of the governing parameters.

INTRODUCTION

EXISTING numerical solutions of two-dimensional heat conduction problems with a moving phase-change interface have traditionally been treated using finite-difference or finite-element methods. These techniques are very time consuming, even with the latest generation of computers, since they require a coupling of the solution for the entire temperature field in the liquid and solid regions with a non-linear energy interface condition at each time increment. The extension of these techniques to three-dimensional problems or problems with embedded fluid-carrying tubes, coupling the heat transfer between the internal fluid and the surrounding phase-change medium, would be extremely costly. In the present paper a new numerical solution procedure is examined for moving phase-change boundary problems in heat transfer based on the boundary integral equation technique. The method has been extensively used in the mechanics of solids where one wishes to determine the quasi-steady stress distribution in the vicinity of a fracture tip [1–3] and more recently in low Reynolds number flow problems with deformable fluid–fluid interfaces [4–6]. The present paper will demonstrate the feasibility of the technique for heat transfer problems in two-dimensions where a quasi-steady approximation is assumed for the temperature distribution in both the liquid and solid regions. The method is currently being extended to three-dimensional problems involving buried tubes with axial thermal interaction and fully transient problems using a time-dependent Green's function.

The two fundamental simplifying features of the boundary integral equation approach that make it especially attractive for heat transfer problems with a moving phase-change interface are: (1) the number of dimensions over which an unknown temperature or its normal gradient must be determined are reduced by one since these unknown distributions need be determined only over the boundaries of each region instead of the areas; and (2) the numerical solution procedure directly provides the unknown normal temperature gradient on each side of the moving interface. This second feature is extremely convenient in the present heat transfer application since the energy interface condition for the instantaneous velocity of the interface involves only the difference in the normal temperature gradients across this surface. The theory for the boundary integral equation approach derives from the conditions of uniqueness for the governing field equations in each region. The solution procedure can be easily applied provided these field equations are linear and possess a relatively simple closed-form fundamental solution which satisfies appropriate boundedness conditions at infinity. Subject to additional constraints of boundary smoothness, these conditions are normally satisfied for both two- and three-dimensional, steady and unsteady heat conduction problems. The new solution approach will be illustrated for the classic problem of a buried pipe in a semi-infinite domain where the freezing/melting is initiated at the planar free surface. This problem has been selected since its solution has not previously appeared in the literature and the dramatic changes in interface shape that evolve as the phase-change boundary gradually wraps around the tube provide a severe test of the flexibility of the solution methodology for handling interfaces of highly irregular shape.

While the boundary integral equation technique has been extensively used in solid mechanics and flow

NOMENCLATURE

b	depth of bypass contour
D	domain
∂D_1	part of boundary where θ is defined
∂D_2	part of boundary where $\partial\theta/\partial n$ is defined
$f(x')$	function of boundary values at x'
$f(\tilde{\xi})$	function of boundary values at $\tilde{\xi}$
f_1, f_2	values of f at the two nodes of an element
G	the Green's function
$g(x')$	function of boundary values at x'
$g(\tilde{\xi})$	function of boundary values at $\tilde{\xi}$
g_1, g_2	values of g at the two nodes of an element
H	coefficient matrix of f
h	depth of the center of the pipe
K	thermal conductivity
k	coefficient matrix of g
l	latent heat
$n(x)$	outward unit normal at x
N	total number of elements
r_{ij}	distance between the centerpoints of the ith and jth elements
$r(\tilde{x}, \tilde{\xi})$	distance between the two points $\tilde{x}$ and $\tilde{\xi}$
R	pipe radius
S	coordinate along the boundary
ΔS	length of the boundary element
T	temperature
T_0	free surface temperature
T_i	pipe temperature
t	time
v_n	normal interface velocity
x, y	Cartesian coordinates
$\tilde{x}$	coordinates of a point (x, y).

Greek symbols

α	angle between normal and x-axis
γ	dimensionless coordinate of an element
∇^2	Laplacian operator
δ	delta function
ε	infinitesimal radius of sphere
ζ_1, ζ_2	interpolation functions
η	coordinate of $\tilde{\xi}$
$\theta(\tilde{x})$	dimensionless temperature function of x
$\theta_0(\tilde{x})$	dimensionless temperature function of the free surface
ξ	moving boundary point (ξ, η)
$\tilde{\xi}$	coordinate of $\tilde{\xi}$
ρ	mass density
τ	dimensionless time
τ_0	dimensionless time required for the interface to intersect the y-axis
τ_s	dimensionless time required for the interface to reach 90% of its steady-state position above the pipe
Ω	region of solid or liquid
$\partial\Omega$	boundary of the solid or liquid region
$\partial\Omega_f$	boundary of the interface.

Superscripts

$*$	dimensionless
$'$	boundary points.

Subscripts

s	solid
L	liquid
$\tilde{}$	coordinates of the points
i	ith boundary element or initial condition
j	jth boundary element.

problems [7, 8], these applications for the most part have been either steady or quasi-steady in nature where one wishes to determine an unknown quasi-steady stress or velocity distribution at the boundaries. In the present situation the solution is required of a non-linear, time-dependent initial value problem in which only the energy matching condition between regions is non-linear. This type of problem is much easier to treat than a moving boundary value problem in which the governing field equations are non-linear. Even with this simplification the success of the method is limited by the rapidity with which the unknown boundary functions can be evaluated in each time interval. For interfaces undergoing large amplitude deformations, as is the present case, the temporal evolvement of the interface may involve a thousand or more time increments. The time-dependent applications of the boundary integral equation technique, which first attracted the authors' attention and suggested the feasibility of this approach for heat transfer applications, were the recent numerical studies of Lee and Leal [5, 6] for the motion of a solid sphere through a deformable fluid–fluid interface. This application from the chemical engineering literature exhibited several of the same essential features of the class of heat transfer problems considered herein: linear field equations in each region, a non-linear matching interface condition and a large amplitude deformation of this interface as time evolves.

HEAT TRANSFER BACKGROUND

Investigators studying the transient behavior of phase-change material (PCM) outside an embedded circular tube have commonly simplified the problem by assuming the PCM to be infinite in extent. With this approximation the problem becomes one-dimensional and its mathematical treatment is

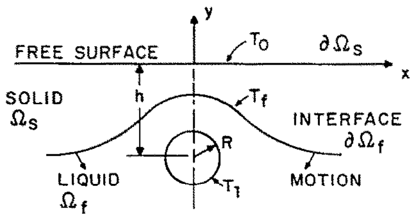

FIG. 1. Schematic of freezing around a buried pipe.

considerably simplified. An important exception where a bounded domain is treated is a tube buried in the vicinity of a planar boundary as shown in Fig. 1. The presence of the planar surface results in a two-dimensional solid–liquid interface motion. Two different cases are identified with this geometry. In the first case the interface commences at the planar surface due to a sudden change in surface conditions and progresses inwards through the PCM while in the second case the interface begins at the pipe surface due to a change in pipe inlet conditions and moves outwards. Thus the initial shape of the interface for the first case is planar while that for the second is cylindrical. Because of the non-linearity of the interface energy equation these two initial value problems are mathematically distinct, having two different solutions. However, they share a common well-known steady-state solution characterized by a circular interface [9] which is eccentrically located about the pipe. To our knowledge the first case, which finds applications in solar collectors, ground freezing and cryoprobe skin treatment in the vicinity of a major blood vessel, has not been treated in the published literature. However, Lunardini [10] recently addressed this case and presented an approximate analytic model for the initial stages of this problem. Applications to buried gas and oil pipes in permafrost have generated considerable interest in the second case. Of importance here is the determination of the outward advance of the thaw front around a warm pipe. Numerical solutions based on variational and finite-element methods to the problem of transient, two-dimensional melting outside a buried pipe at uniform surface temperature have been obtained for various planar surface boundary conditions by Lachenbruch [11], Gold *et al.* [12], Wheeler [13] and Hwang [14]. Because these solutions are expensive to generate, several investigators have obtained various approximate analytic solutions to both insulated and bare buried pipes [15–23]. Without exception, all these two-dimensional analytic solutions are based on a simplified quasi-steady approximation in which the transient term in the heat conduction equation is neglected in both phases. This is a valid approximation when the latent heat of the PCM is large compared to the sensible heat (i.e. small Stefan number).

In this paper we examine the first case where the interface commences at the planar surface. The semi-infinite PCM is assumed to be initially at uniform temperature T_i which is above its melting temperature T_f. The planar surface temperature of the PCM is suddenly changed to T_0 where $T_0 < T_f < T_i$. A tube of radius R and surface temperature T_i is buried a distance h below the planar surface.

Mathematical formulation

In the problem described above and illustrated in Fig. 1 the solid domain is defined by Ω_s with its boundaries described by $\partial\Omega_s$ (free surface) and $\partial\Omega_f$ (interface). The liquid domain is defined by Ω_L with its boundaries described by $\partial\Omega_L$ (pipe surface) and $\partial\Omega_f$. The governing differential equation for quasi-steady two-dimensional conduction are

$$\nabla^2 T_s(\underset{\sim}{x}) = 0, \quad \forall \underset{\sim}{x} \in \Omega_s \tag{1a}$$

$$\nabla^2 T_L(\underset{\sim}{x}) = 0, \quad \forall \underset{\sim}{x} \in \Omega_L \tag{1b}$$

where T_s and T_L are the temperatures of the solid and liquid regions respectively, ∇^2 is the Laplacian operator and $\underset{\sim}{x} = (x, y)$. The boundary conditions are

$$T_s(\underset{\sim}{x}', t) = T_0, \quad \forall \underset{\sim}{x}' \in \partial\Omega_s \tag{2a}$$

$$T_s(\underset{\sim}{x}', t) = T_f, \quad \forall \underset{\sim}{x}' \in \partial\Omega_f \tag{2b}$$

and

$$T_L(\underset{\sim}{x}', t) = T_f, \quad \forall \underset{\sim}{x}' \in \partial\Omega_f \tag{3a}$$

$$T_L(\underset{\sim}{x}', t) = T_i, \quad \forall \underset{\sim}{x}' \in \partial\Omega_L. \tag{3b}$$

The energy balance at the interface requires

$$K_s \frac{\partial T_s(\underset{\sim}{x}')}{\partial n_s(\underset{\sim}{x}')} - K_L \frac{\partial T_L(\underset{\sim}{x}')}{\partial n_L(\underset{\sim}{x}')} = \rho l v_n(\underset{\sim}{x}'), \quad \forall \underset{\sim}{x}' \in \partial\Omega_f \tag{4}$$

where $n(\underset{\sim}{x}')$ is the outward normal coordinate at the interface, K_s and K_L are the thermal conductivity of solid and liquid, respectively, ρ is the mass density, l is the latent heat and $v_n(\underset{\sim}{x}') = \mathrm{d}n(\underset{\sim}{x}')/\mathrm{d}t$ is the interface velocity of $\underset{\sim}{x}'$ in the normal direction.

To non-dimensionalize the problem the following dimensionless variables are introduced:

$$\theta_s(\underset{\sim}{x}) = \frac{K_s}{K_L}\frac{T_s(\underset{\sim}{x}) - T_f}{T_i - T_f} \tag{5}$$

$$\theta_L(\underset{\sim}{x}) = \frac{T_L(\underset{\sim}{x}) - T_f}{T_i - T_f} \tag{6}$$

$$h^* = \frac{h}{R}, \quad n^*(\underset{\sim}{x}) = \frac{n(\underset{\sim}{x})}{R}, \quad \underset{\sim}{x} = \frac{x}{R} \tag{7}$$

$$\tau = \frac{K_L(T_i - T_f)}{R^2 \rho l} t \tag{8}$$

where t is time and τ is the dimensionless time.

The dimensionless form of the governing equations, the boundary conditions and the energy balance at the interface are:

$$\nabla^2 \theta_s(\underset{\sim}{x}) = 0, \quad \forall \underset{\sim}{x} \in \Omega_s \tag{9a}$$

$$\nabla^2 \theta_L(\underset{\sim}{x}) = 0, \quad \forall \underset{\sim}{x} \in \Omega_L \tag{9b}$$

$$\theta_s = \theta_0 \quad \text{on } \partial\Omega_s \tag{10a}$$

$$\theta_s = 0 \quad \text{on } \partial\Omega_f \tag{10b}$$

$$\theta_L = 0 \quad \text{on } \partial\Omega_f \tag{11a}$$

$$\theta_L = 1 \quad \text{on } \partial\Omega_L \tag{11b}$$

and

$$\frac{\partial\theta_s(x')}{\partial n_s^*(x')} - \frac{\partial\theta_L(x')}{\partial n_L^*(x')} = \frac{dn_s^*(x')}{d\tau}, \quad x' \in \partial\Omega_f \tag{12}$$

where $\theta_0 = K_s(T_0 - T_f)/K_L(T_i - T_f)$.

The governing equations and boundary conditions show that the physical problem just described is completely defined by only two dimensionless groups if the sensible heat of the liquid and solid phases are neglected compared to the heat of fusion: a geometric parameter $h^* = h/R$ and a temperature ratio θ_0. We wish to examine the behavior of the interface as these two parameters are varied.

THE BOUNDARY INTEGRAL EQUATION

For a general, two- or three-dimensional problem, let $u(x)$ be the solution of the harmonic equation

$$\nabla^2\theta(x) = 0, \quad x \in D \tag{13}$$

subject to the boundary conditions

$$\theta(x') = f(x'), \quad \forall x' \in \partial D_1 \tag{14}$$

$$\frac{\partial\theta(x')}{\partial n(x')} = g(x'), \quad \forall x' \in \partial D_2 \tag{15}$$

where f and g are specified boundary values and $\partial D = \partial D_1 + \partial D_2$. Let $G(x, \xi)$ be the Green's function, i.e. the solution of $\nabla^2 G(x, \xi) = \delta(x-\xi)$ where δ is the delta function. Employing the Green's second identity, which is

$$\int_D [\theta(x)\nabla^2 G(x,\xi) - G(x,\xi)\nabla^2\theta(x)]\, dD$$
$$= \int_{\partial D}\left[\theta(\xi)\frac{\partial G(x,\xi)}{\partial n(\xi)} - G(x,\xi)\frac{\partial\theta(\xi)}{\partial n(\xi)}\right] dS(\xi) \tag{16}$$

one can readily arrive at

$$\theta(x) = \int_{\partial D}\left[f(\xi)\frac{\partial G(x,\xi)}{\partial n(\xi)} - g(\xi)G(x,\xi)\right] dS(\xi). \tag{17}$$

Let x approach a boundary point x', then equation (17) becomes

$$f(x') = \int_{\partial D} f(\xi)\frac{\partial G(x',\xi)}{\partial n(\xi)}\, dS(\xi) - \int_{\partial D} g(\xi)G(x',\xi)\, dS(\xi). \tag{18}$$

As ξ approaches x', the first integral on the LHS of equation (18) becomes singular. However, this singularity is avoided by following a different path such as a small spherical region of radius ε about the singular point, i.e. $x' = \xi$, Fig. 2. Taking the limit as $\varepsilon \to 0$, the integral exhibits a jump of $f(x')/2$. Equation

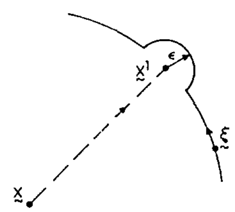

FIG. 2. Alternative path when $x' = \xi$.

(18) becomes

$$\tfrac{1}{2}f(x') + \int_{\partial D} G(x',\xi)g(\xi)\, dS(\xi) = \oint_{x' \neq \xi} f(\xi)\frac{\partial G(x',\xi)}{\partial n(\xi)}\, dS(\xi). \tag{19}$$

Depending upon the first or the second boundary condition, (19) becomes the standard Fredholm equation of the first or the second kind, respectively. Once f and g at any point are known, the solution at any point in the domain will be obtained from equation (17). For the two-dimensional problem considered here, the Green's function is $G(x', \xi) = (1/2\pi)\ln r(x', \xi)$, where r denotes the distance of point x' from the point ξ. Therefore, as ξ approaches x', the integral on the LHS of (19) becomes singular. To avoid the singularity we make the assumption that $g(\xi)$ is constant over a small interval ΔS of the boundary and integrate over the path shown in Fig. 3. Taking the limit as b approaches zero, equation (19) can be written as:

$$\tfrac{1}{2}f(x') + \frac{1}{2\pi}g(x')\Delta S\left(\ln\left|\frac{\Delta S}{2}\right| - 1\right) + \frac{1}{2\pi}\oint_{\partial D, x' \neq \xi} g(\xi)\ln r(x',\xi)\, dS(\xi)$$
$$= \frac{1}{2\pi}\oint_{\partial D, x' \neq \xi} f(\xi)\left[-\frac{x-\xi}{r^2}\cos\alpha(\xi) - \frac{y-\eta}{r^2}\sin\alpha(\xi)\right] dS(\xi) \tag{20}$$

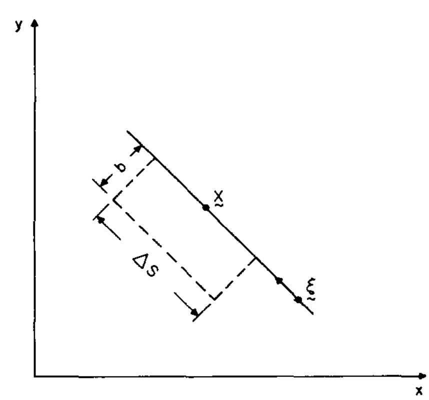

FIG. 3. The integration contour.

where $\underset{\sim}{x}' = (x, y)$, $\underset{\sim}{\xi} = (\xi, \eta)$, $r^2 = (x-\xi)^2+(y-\eta)^2$ and $\alpha(\underset{\sim}{\xi})$ is the angle between the outward normal and x axis at the boundary point.

To obtain a numerical solution to (20) we subdivide the boundary ∂D into $N+1$ nodes and N discrete intervals, boundary elements, ΔS_i ($i = 1, \ldots, N$). The intervals can be considered as constant, linear, quadratic or higher-order boundary elements. Clearly the shape function will be different for each type of element. For a constant element, $f(\underset{\sim}{x}')$ and $g(\underset{\sim}{x}')$ are constant over each interval and can be taken out of the integrals. Therefore, equation (20) leads to

$$\frac{1}{2\pi} g_i \Delta S_i \left(\ln \left| \frac{\Delta S_i}{2} \right| - 1 \right) + \frac{1}{2\pi} \sum_{\substack{j=1 \\ j \neq i}}^{N} \left[\int_{\Delta S_j} \ln r_{ij} \, \mathrm{d}S_j \right] g_j$$

$$= -\tfrac{1}{2} f_i + \frac{1}{2\pi} \sum_{\substack{j=1 \\ j \neq i}}^{N}$$

$$\left[\int_{\Delta S_j} -\frac{(x_i - x_j)(x_j - x_{j+1}) + (y_i - y_j)(y_{j+1} - y_j)}{r^2 \Delta S_j} \, \mathrm{d}S_j \right] f_j. \tag{21}$$

In the above equation the subscripts i and j represent the value at the mid-point of the intervals and (x_i, y_i) and (x_j, y_j) denote the coordinates of the points $\underset{\sim}{x}'$ and $\underset{\sim}{\xi}$, respectively.

For a linear boundary element the values of $f(\underset{\sim}{x}')$ and $g(\underset{\sim}{x}')$ at any point on the element can be defined in terms of their nodal values and two linear interpolation functions ζ_1 and ζ_2

$$f(\gamma) = \zeta_1 f_1 + \zeta_2 f_2$$

$$g(\gamma) = \zeta_1 g_1 + \zeta_2 g_2$$

where f_1, g_1 and f_2, g_2 are the boundary values of two nodes of an element, dimensionless coordinate γ is equal to $\underset{\sim}{x}'/(\Delta S/2)$, ΔS is the length of the element and the ζ_1, ζ_2 functions are given by

$$\zeta_1 = \tfrac{1}{2}(1-\gamma), \quad \zeta_2 = \tfrac{1}{2}(1+\gamma).$$

Therefore each integral of equation (20) will have two parts associated with the interpolation functions ζ_1 and ζ_2. In either case (20) and/or (21) can be written in the following matrix form

$$[K_{ij}]\{g_i\} = [H_{ij}]\{f_i\} = \{F_i\} \tag{22}$$

where $[K_{ij}]$ is an $N \times N$ matrix whose elements contain the boundary integral of the Green's function, $\{g_i\}$ is an $N \times 1$ vector whose elements contain the unknown normal derivative of the dimensionless temperature $\partial\theta/\partial n$, $[H_{ij}]$ is an $N \times N$ matrix whose elements contain the coefficient of dimensionless temperature and $\{f_i\}$ is an $N \times 1$ vector whose elements contain the known value of the dimensionless temperature. The RHS is known and is denoted by $\{F_i\}$. Once equation (22) is solved for the vector $\{g_i\}$, the temperature at any point in the domain $\underset{\sim}{x} = (x, y)$ is given by

$$\theta(\underset{\sim}{x}) = \frac{1}{2\pi} \sum_{i=1}^{N}$$

$$\left[\int_{\Delta S_i} -\frac{(x-\xi_i)(\xi_i - \xi_{i+1}) + (y - \eta_i)(\eta_{j+1} - \eta_j)}{r^2(x, \xi_i)\Delta S_i} \, \mathrm{d}S_i \right]$$

$$f_i - \frac{1}{2\pi} \sum_{i=1}^{N} \left[\int_{\Delta S_i} \ln r(x, \xi_i) \, \mathrm{d}S_i \right] g_i \tag{23}$$

where $\xi_i = (\xi_i, \eta_i)$ is a boundary point at subdivision i. In order to determine the velocity and the position of the interface, equation (22) is first solved for the solid and liquid regions shown in Fig. 1. The energy equation (12) is then used to determine n^* which leads to the determination of the velocity and the position of the interface. The traveling direction and the direction of the outward unit normal n^* on the boundaries of the solid and liquid regions are defined in Fig. 4. As shown in Fig. 4 the outward unit normals at the interface of the two regions are in opposite directions, therefore equation (12) becomes

$$\frac{\partial \theta_{si}}{\partial n_{si}^*} + \frac{\partial \theta_{Li}}{\partial n_{Li}^*} = \frac{\mathrm{d}n_i^*}{\mathrm{d}\tau} \tag{24}$$

where θ_{si} and θ_{Li} are the dimensionless temperature of solid and liquid at the centerpoint of the ith element, respectively, n_{si}^* and n_{Li}^* are the two normals at the centerpoint of the ith element of solid and liquid, respectively, and n_i^* is the unit normal of the centerpoint of the ith element of the interface.

Computations

Due to the symmetry of the problem with respect to the y-axis, half of the domain is sufficient for the discretization. This will clearly decrease the computation time by a factor of approximately 2. To avoid the effect of the pipe on the far field solution, point A in Fig. 4, the starting node is chosen at least 30 pipe radii from the origin. The straight line part of the boundary, the interface boundary and the semi-circular boundaries are each subdivided into 12 unequal meshes. After a few numerical calculations, it

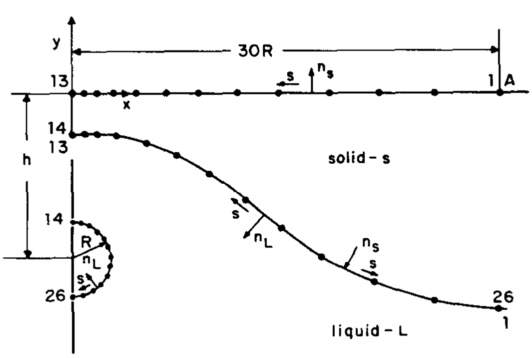

FIG. 4. Elements arrangement.

was found that 12 meshes on each side of the boundary are an adequate number of meshes for good accuracy and computing efficiency. The dimensions and the boundary elements are depicted in Fig. 4. The moving velocity for each mesh, which is found by (24) is used to determine the new location of the nodes on the interface. For better accuracy, however, as the change in curvature of the interface becomes larger, the nodes are slightly rearranged and the length of the boundary elements are changed to represent a more accurate interface movement.

To start the numerical computation Neumann's solution [24] is used to generate a thin solid-phase region and to estimate the first step of the interface motion.

The thickness of the starting solid phase generated by Neumann's solution is so small ($10^{-2} \times R$) that the boundary integral equation is not suitable for the solid region. This is due to the fact that the solution of the boundary integral equation is not sufficiently accurate for very thin regions that have thickness of the order of $10^{-2} \times$ (element length). Therefore, for the next few time steps, we adapt a quasi-linear temperature distribution in the solid region until the thickness of the region reaches the order of 10^{-1}. Thus the normal derivative of the temperature of the solid region at the interface can be written as

$$\frac{\partial T_s(x')}{\partial n_s(x')} = \frac{T_0 - T_f}{y_i} \frac{1}{\sqrt{1+(dy_i/dx_i)^2}} \tag{25}$$

and the nondimensionalized form is given by

$$\frac{\partial \theta_s(x')}{\partial n_s(x')} = \frac{K_s}{K_L}\left(\frac{T_0 - T_f}{T_i - T_f}\right)\frac{R}{y_i}\frac{1}{\sqrt{1+(dy_i/dx_i)^2}} \tag{26}$$

where $x' = (x_i, y_i)$.

RESULTS

The transient interface location for freezing around a pipe buried at a distance $h^* = -3$ below a free surface is shown in Fig. 5. The interface commences at the surface, which is maintained at $\theta_0 = -1$, and proceeds towards the pipe. In the region directly above the pipe the interface moves rapidly towards its steady state while it continues to undergo significant development elsewhere. At $\tau = \tau_0 = 402.9$ the interface intersects the vertical line of symmetry and forms a closed loop around the pipe. Subsequently two fronts are formed: a cylindrical interface which encloses the pipe and moves towards it, and a curved interface which progresses away from the pipe. At steady state (not shown) the former becomes circular while the latter planar. In Fig. 5 the numerical computation is terminated at $\tau = \tau_0$. However, extension of the solution for $\tau > \tau_0$ does not present any conceptual difficulties. These features are preserved when the surface temperature is lowered to $\theta_0 = -4$ as shown in Fig. 6. The interface for this case progresses much more rapidly than that for $\theta_0 = -1$, intersecting the vertical axis below the pipe at $\tau_0 = 22.08$. The variation of τ_0 with the dimensionless surface temperature for $h^* = -3$ is shown in Fig. 7. The effect of pipe depth h^* on τ_0 for $\theta_0 = -1$ and $\theta_0 = -4$ is presented in Fig. 8. As might be expected an increase in the pipe depth h^* results in an increase in the time required for the interface to enclose the pipe.

Turning now to the behavior of the interface in the region directly above the pipe, we examine the time needed for the interface to approach its steady-state position. Since steady state is reached asymptotically as $\tau \to \infty$, we arbitrarily define τ_s as the dimensionless time required for the interface to reach 90% of its steady-state position above the pipe at $x = 0$. Defining

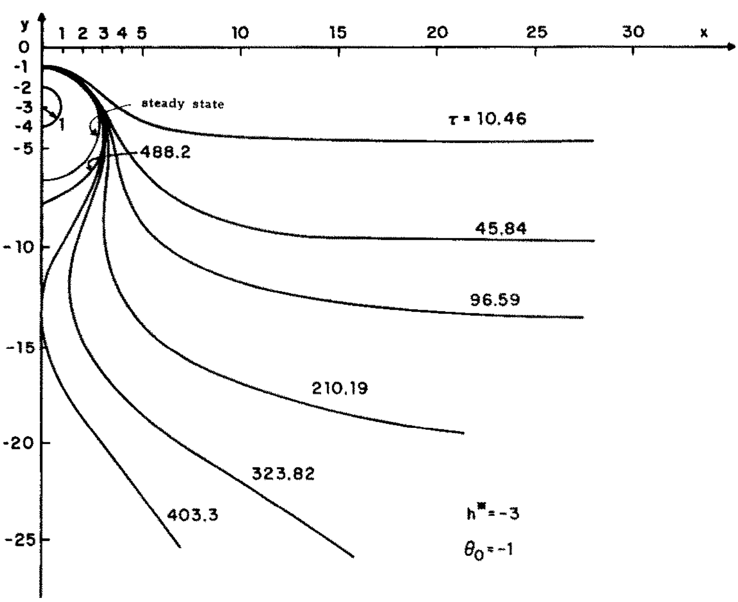

FIG. 5. Interface motion, $\theta_0 = -1$, $h^* = -3$.

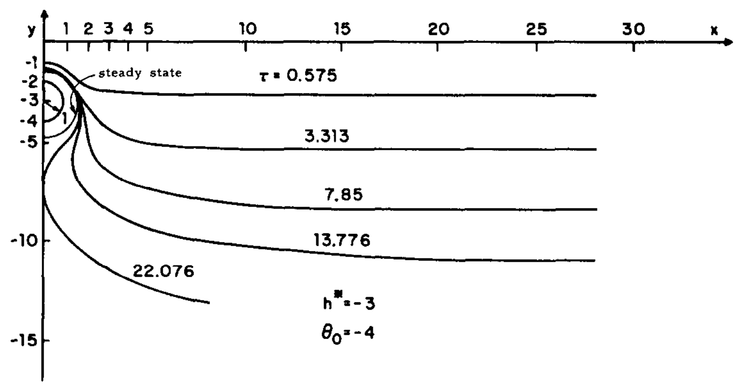

FIG. 6. Interface motion, $\theta_0 = -4$, $h^* = -3$.

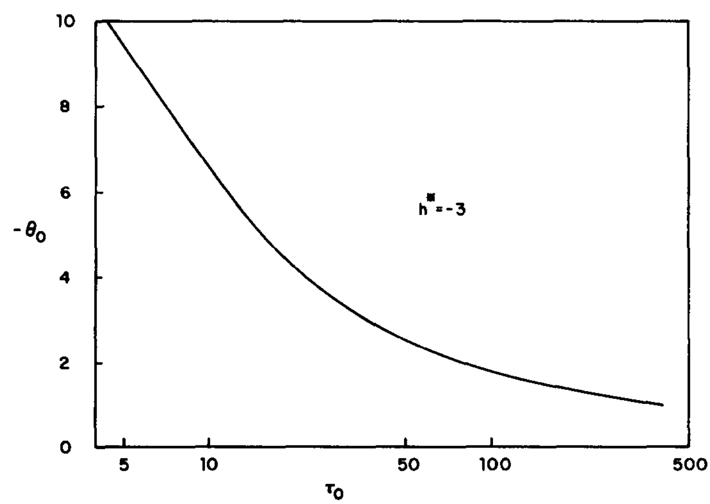

FIG. 7. Surface temperature effect on the time for the interface to enclose pipe.

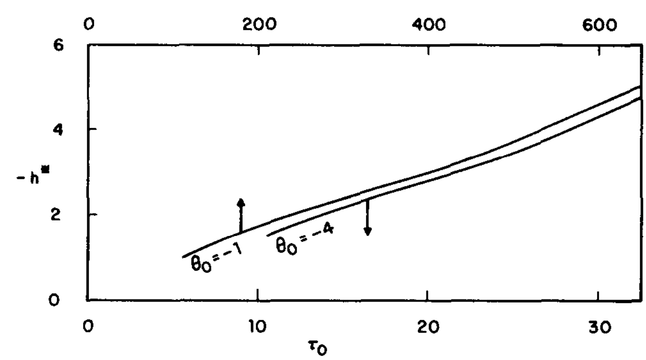

FIG. 8. Pipe depth and surface temperature effect on time needed for the interface to enclose pipe.

$\delta(\tau)$ as the interface location at $x = 0$ and $\delta(\infty)$ as the steady-state location, it follows that τ_s is the dimensionless time corresponding to $\delta(\tau_s)/\delta(\infty) = 0.9$. The effect of θ_0 on τ_s is shown in Fig. 9 for $h^* = -3$. Here we note a significant drop in τ_s as the dimensionless surface temperature θ_0 is decreased. The effect of pipe depth on τ_s is illustrated in Fig. 10 for $\theta_0 = -1$ and -4.

Discussion

The accuracy of the solution was examined with regard to the number and size of elements as well as the extent of the integration domain. Each of the three boundaries shown in Fig. 4, the planar surface, interface and pipe surface, was divided into unequal meshes. The results presented in Figs. 5–10 are based on a network of 12 elements for each boundary.

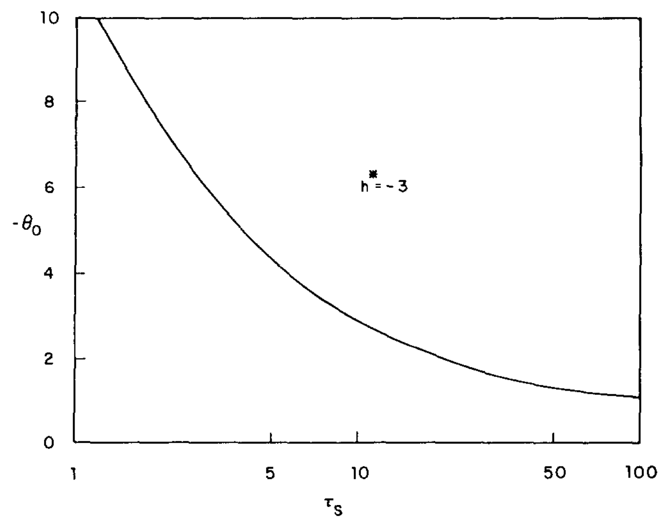

FIG. 9. Surface temperature effect on time needed for the interface to reach 90% of its steady-state location above the pipe.

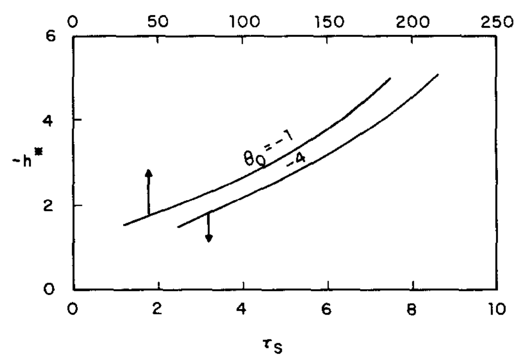

FIG. 10. Effect of pipe depth on time needed for the interface to reach 90% of its steady-state location above the pipe.

Increasing the number of elements to 18 and 24 was found to increase computational time by a factor of two and four, respectively but did not change the results sufficiently to justify increasing the number of elements beyond 12. On the other hand, decreasing the number of elements to eight or less led to an unacceptable result. For the free surface and the interface, the size of the elements close to the y-axis was made smaller than those far away. Furthermore, since the interface curves as it moves, care was taken to use small size elements along the interface where the curvature is large. Another factor which affects the solution is the extent of the integration domain in the x-direction as defined by location A in Fig. 4. Based on several numerical calculations it was found that if the location of A is increased beyond 30 pipe radii from the vertical axis, little change in the results will be observed. On the other hand a distance of 20 pipe radii was found to give unacceptable results.

The computational time was found to be dependent on the surface temperature θ_0. For $\theta_0 = -1$, 408 s on the IBM 3033 were needed for the interface to envelope the pipe. The corresponding time for $\theta_0 = -4$ is 33 s.

The solution far away from the pipe was compared with the quasi-steady, one-dimensional analytic solution to Neumann's problem. Good agreement was observed for the cases presented in Figs. 5 and 6. However, when the effect of sensible heat is taken into consideration the interface position given by Neumann's solution was found to lag behind the quasi-steady solution. It is therefore expected that the quasi-steady approximation results in an overestimate of the interface advance for the two-dimensional problem considered.

The use of the boundary integral method to solve two-dimensional phase-change problems as illustrated by the buried pipe example with the interface commencing at the free surface can be easily used to treat the corresponding problem where the interface starts at the pipe surface.

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TECHNIQUE DE L'EQUATION INTEGRALE DE LIMITE AVEC APPLICATION AU GEL AUTOUR D'UN TUBE ENTERRE

Résumé—On considère l'utilisation de la méthode de l'équation intégrale de limite (BIEM) pour des problèmes multidimensionnels avec interface mobile entre phases. La méthode est trouvée adaptée aux problèmes de transfert thermique dans lesquels les équations de champ sont linéaires dans chaque région et les conditions aux limites ou à l'interface sont à la fois fortement mobile, la technique BIE réduit d'une unité les dimensions du problème, ce qui diminue les opérations de mise en mémoire, et elle résout directement pour le gradient de température normal inconnu de chaque coté de l'interface et pour la vitesse instantanée de l'interface. Pour illustrer la souplesse de cette technique la méthode IEM est appliquée à un problème actuellement non résolu: la fusion ou le gel autour d'un tube enterré dans un domaine semi-infini où la fusion ou gel sont initiés à la surface libre et où le milieu n'est pas initialement à la température de changement de phase. Pour simplifier on suppose une conduction thermique stationnaire dans les deux zones. Des solutions sont présentées pour plusieurs valeurs des paramètres actifs.

ANWENDUNG DER GRENZ-INTEGRAL-GLEICHUNGS-METHODE AUF GEFRIERVORGÄNGE UM EIN EINGEGRABENES ROHR

Zusammenfassung—In dieser Arbeit wird die Anwendung der Grenz-Integral-Gleichungs-Methode (BIEM) auf mehrdimensionale Probleme mit variabler Phasengrenzfläche untersucht. Es zigt sich, daß die Methode für Wärmetransportprobleme geeignet ist, bei denen die Gleichungen in jedem Gebeit linear sind, die Rand- oder Phasengrenzflächenbedingungen jedoch in hohem Maße ungleichmäßig und nichtlinear sein dürfen. Für Probleme mit beweglicher Phasengrenzfläche reduziert BIEM einerseits den Grad des Problems um Eins, wodurch weniger Speicherplatz benötigt wird; das Verfahren ermittelt andererseits für beide Seiten der Phasengrenzfläche den unbekannten Temperaturgradienten in Normalen-Richtung, der zur Bestimmung der Momentan-Geschwindigkeit der Phasengrenzfläche verwendet wird. Um die Vielseitigkeit des Verfahrens zu demonstrieren, wird BIEM auf ein bisher ungelöstes Problem angewandt: Den Schmelz- und Gefriervorgang um ein eingegrabenes Rohr in einer halbunendlichen Umgebung, bei dem das Schmelzen bzw. Gefrieren an der freien Oberfläche eingeleitet wird und das Medium anfangs nicht auf Phasenwechseltemperatur ist. Der Einfachheit halber wird quasistationäre Wärmeleitung in der ungefrorenen und gefrorenen Zone angenommen. Es werden Ergebnisse für die verschiedenen Hauptparameter des Problems vorgestellt.

ИСПОЛЬЗОВАНИЕ МЕТОДА ГРАНИЧНОГО ИНТЕГРАЛЬНОГО УРАВНЕНИЯ ДЛЯ ОПИСАНИЯ ПРОЦЕССА ЗАМЕРЗАНИЯ СРЕДЫ ВОКРУГ ПОГРУЖЕННОЙ ТРУБЫ

Аннотация—Используется метод граничного интегрального уравнения для решения многомерных задач с подвижной границей раздела фаз. Показано, что метод можно использовать для решения задач теплопереноса, когда уравнения для полей являются линейными в каждой из областей, а граничные условия, или условия сопряжения фаз, являются как нерегулярными, так и нелинейными. Для задач с подвижной фазовой границей предлагаемый метод позволяет как сократить размерность задачи на одну единицу, что уменьшает объем необходимой для решения информации, так и получить прямое решение для неизвестного температурного градиента по нормали с каждой стороны границы раздела для определения на ней текущего значения скорости. Для иллюстрации возможностей метода дано решение ранее нерешенной задачи: таяние/замерзание среды вокруг трубы, погруженной в полубесконечную область, когда процесс начинается на свободной поверхности и температура среды вначале отлична от температуры фазового перехода. Для упрощения задачи делается предположение о квази-стационарном характере процесса теплопроводности как в растаявшей, так и в замерзшей зонах. Решения даны для различных значений основных параметров.